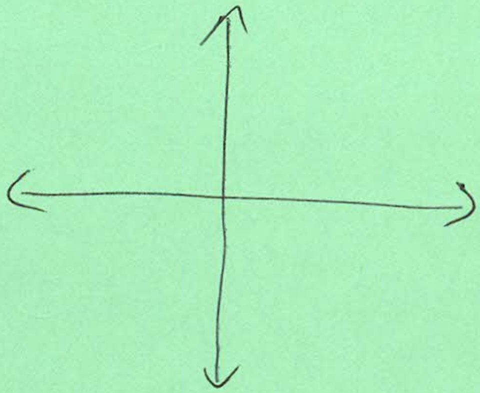
Surfaces

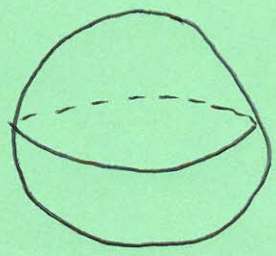
A <u>surface</u> is a 2-dimensional topological space.

Examples include:

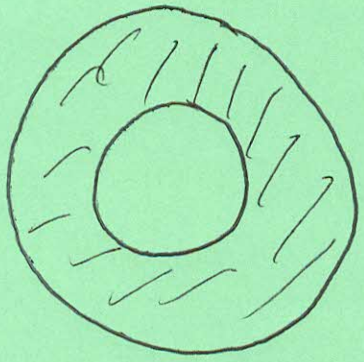
ⓐ The whole xy-plane ($\mathbb{R}^2$)



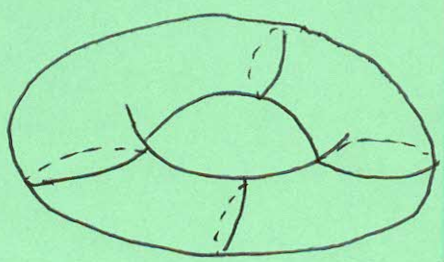
ⓑ The (hollow) Sphere (S^2)



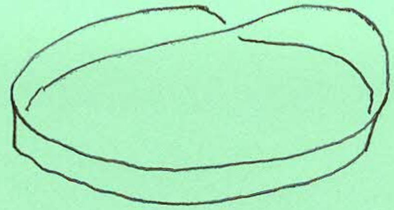
ⓒ The annulus



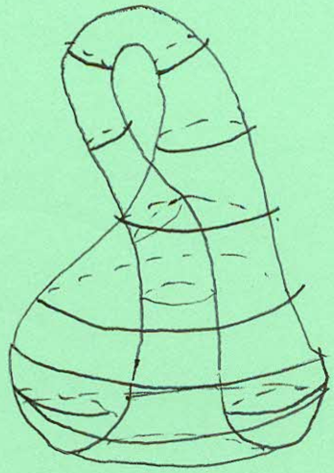
ⓓ The (hollow) Torus ($\mathbb{T}$ [illegible])



(e) Möbius Band



(f) Klein Bottle ($\mathbb{K}$)



These surfaces can be classified according to two criteria: closedness & orientability.

The easier of the two is whether a surface is closed.

Def: A surface is closed if it has no boundary and a (possibly very large) ball can be drawn around it.

A boundary of a surface is a part of the surface you cannot walk past.

Examples with boundary: c, e

Examples without boundary: a, b, d, f

Of the surfaces above only a does not fit into a ball. Thus, we can split the surfaces into 2 groups:

Closed: b, d, f ; Not-closed: a, c, e

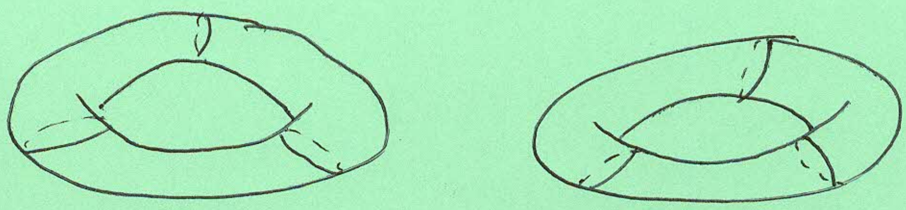
We'll hold off on orientability for a moment. Also, to make our lives simpler, we will only deal with closed surfaces.

Let's now describe how to create a new surface out of two old ones. This procedure is called the connected sum. To form this, we cut out disks in each surface, then connect them with a tube. Let's do an example. The notation for the connected sum of two surfaces M & N is

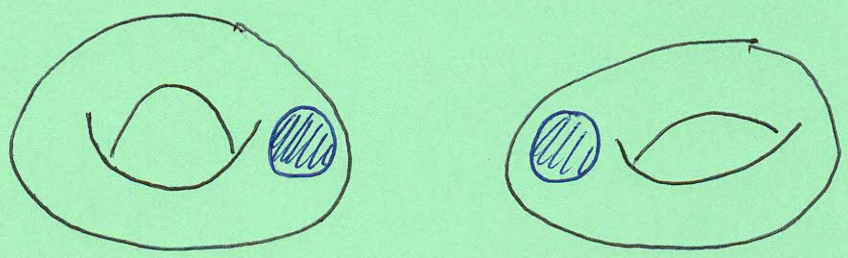
$$\boxed{M \# N}$$

Ex: What is $\mathbb{T} \# \mathbb{T}$?

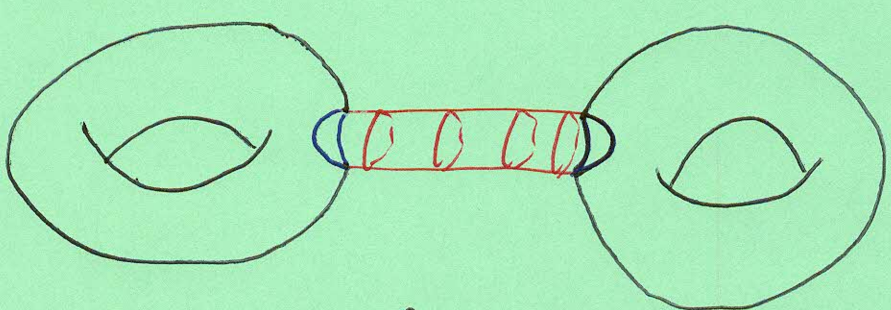
Sol: Start with two copies of $\mathbb{T}$:



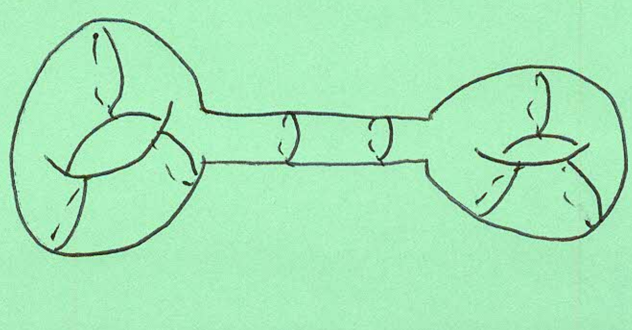
Cut a hole in each one:



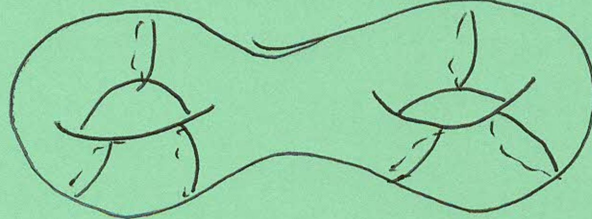
Attach one end of a tube (cylinder) to each surface at the hole:



We get a surface which looks like this

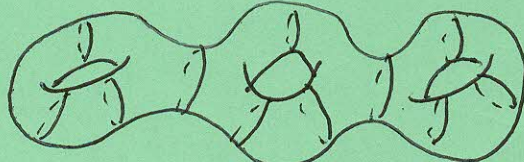


Which is the same as

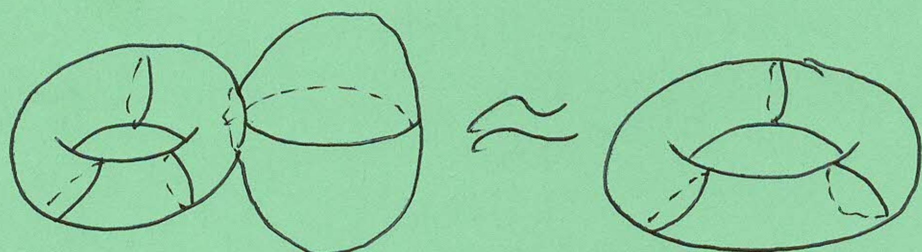


We will call this surface $2\mathbb{T}$, and by extension: $n\mathbb{T} = \underbrace{\mathbb{T} \# \cdots \# \mathbb{T}}_{n\text{-times}}$.

Ex: (a) $3\mathbb{T}$:



(b) $\mathbb{T} \# S^2$ $\qquad\qquad$ $\mathbb{T}$



(c) $\mathbb{T} \# S^2 \# \mathbb{T}$ $\qquad\qquad$ $2\mathbb{T}$